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SMALL COMPRESSIBLE MODULES AND SMALL RETRACTABLE MODULES

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ABSTRACT

Let R be a commutative ring with identity, and M be (left) unitary R-module. In this paper we introduce a detailed study for the concepts small compressible modules and small retractable modules.

KEYWORDS: Small Compressible Modules and Small Retractable Modules

1. INTRODUCTION

Let R be a commutative ring with identity and M be (left) unitary R-module. M is called small compressibleif M can be embedded in each of its non-zero small submodule[5]. M is called small retractable if $Hom_R(M,N) \neq 0$ for each non-zero small submodule N of M[5].where aproper submodule N of an R-module M is called small submodule ($N \ll M$) if for any submodule K of M with $N + K = M$ implies $K = M$[4]. We establish the basic properties of these two concepts in details.

The main goal of this research is to study small compressible modules and small retractable modules.

This research consists of three sections. In section two we investigate the basic properties of small compressible modules. In the third section we shall concerned with the basic properties of small retractable modules. Some characterizations of small retractable modules are given in the fourth section.

2. SMALL COMPRESSIBLE MODULES

The concepts of small compressible and small critically compressible modules are introduced in this section and many of their basic properties are studied, moreover we give some characterizations of these concepts.

2.1 Definition[5]:

An R-module M is called small compressibleif M can be embedded in each of its non-zero small submodule.

Equivalently, M is small compressible if there exists a monomorphism from M into N whenever $0 \neq N \ll M$.

A ring R is called small compressible if the R-module R is small compressible. That is R can be embedded in any of its non-zero small ideal.

2.2 Examples and Remarks:

(1) Every compressible module is small compressible and the converse is not true in general, for instance Z_6as a Z-module is not compressible but $Z_6$is small compressible since 0 is only small submodule of Z_6 .

(2) Let$M$ be a small compressible module such that every submodule of M contains a non-zero small submodule of M, then M is compressible.

Proof: Let $0 \neq N \leq M$. By hypothesis there exists $0 \neq K \leq N$, then $K \ll M$ [6]since M is small compressible there exists$0 \neq f: M \to K$is a monomorphism, $if: M \to N$ is a non-zero monomorphism, where $i: K \to N$ be the incluction homomorphism then M is compressible.

(3) The Z-module Q is not small compressible since $Z \ll Q$ and $Hom(Q, Z) = 0$.

(4) Z_4as a Z-module is not small compressible, since $(\bar{2}) \ll Z_4$ but Z_4 cannot be embedded in $(\bar{2})$.

(5) If M is a hollow module (every submodule of M is small in M). Then M is small compressible if and only if M is compressible.

(6) Every simple module is small compressible but not conversely, since Z as a Z-module is small compressible but not simple.

(7) Each of the rings Z and $Z_6$is a small compressible ring.

(8)A module M is small compressible if and only if M can be embedded in Rx for each $0 \neq x \in M$ and $Rx \ll M$.

Proof: (⇒) Is obvious according to the definition (2.1).

(⇐) Let $0 \neq N \ll M$ and let $0 \neq x \in N$.Then $Rx \ll M$[4].By hypothesis there is a monomorphism say, $f: M \to Rx$ so, the composition $M \xrightarrow{f} Rx \xrightarrow{i} N$ is a monomorphism with $i: Rx \to N$ is the inclusion homomorphism. Hence M is small compressible.

(9)A small compressible module M is compressible if every cyclic submodule of M is small in M.

Proof: Let $0 \neq N \leq M$ and $0 \neq x \in N$. Then by hypothesis $Rx \ll M$ so there is a monomorphism $f: M \to Rx$ and hence the composition $M \xrightarrow{f} Rx \xrightarrow{i} N$ is a monomorphism which implies that M is compressible.

(10) Let M be a module in which every cyclic submodule of M is small in M. Then M is compressible if and only if M is small compressible.

2.3 Proposition:

A small submodule of a small compressible module is also small compressible.

Proof: Let M be a small compressible module and $0 \neq N \ll M$. Let $0 \neq K \ll N$. Then $K \ll M$[4]. As M is small compressible implies there exists a monomorphism, say $f: M \to K$ and therefore $fi: N \to K$ is a monomorphism where $i: N \to M$ is the inclusion homomorphism. Hence N is small compressible.

2.4 Proposition:

A direct summand of a small compressible module is also small compressible.

Proof: Let $M = A \oplus B$ be a small compressible module and let $0 \neq K \ll A$. Then $K \oplus 0 \ll M$ [6]and hence there is a monomorphism say, $f: M \to K \oplus 0$ clearly $K \oplus 0 \simeq K$, so $f: M \to K$ is a monomorphism and the composition $A \xrightarrow{j_A} M \xrightarrow{f} K$ is a monomorphism where j_A is the injection of A into M. Therefore A is small compressible.

2.5 Proposition:

Let $M_1$and M_2be two isomorphic modules. Then M_1 is small compressible if and only if $M_2$is small compressible.

Proof: Assume that $M_1$is small compressible and let $\varphi : M_1 \to M_2$ be an isomorphism. Let $0 \neq N \ll M_2$. Then $0 \neq \varphi^{-1}(N) \ll M_1$ .Put $K = \varphi^{-1}(N)$. Let $f: M_1 \to K$be a monomorphism and let$g = \varphi|_K$ then $g: K \to M_2$ is a monomorphism and $g(k) = \varphi(\varphi^{-1}(N)) = N$, hence $g: K \to N$ is a monomorphism. Now, we have the composition $M_2 \xrightarrow{\varphi^{-1}} M_1 \xrightarrow{f} K \xrightarrow{g} N$. Let $h = gf\varphi^{-1}$ is a monomorphism. Therefore M_2 is small compressible.

2.6 Remark:

A homomorphic image of a small compressible module need not be small compressible in general.

For example, *Z* as a *Z*-module is small compressible and $Z/4Z \simeq Z_4$is not small compressible.

2.7 Proposition:

Let $M = M_1 \oplus M_2$be an R-module such that $annM_1 + annM_2 = R$. Then *M* is small compressible if and only if M_1 and $M_2$are small compressible.

Proof: (⇒)Follows fromProposition(2.4).

(⇐) Let $0 \neq N \ll M$. Then by [9],$N = K_1 \oplus K_2$for some $0 \neq K_1 \leq M_1$and $0 \neq K_2 \leq M_2$. And as $N \ll M$, then $K_1 \ll M_1 \leq M$ and $K_2 \ll M_2 \leq M$ by [6] But M_1 and M_2 are small compressible, so there are monomorphisms$f: M_1 \to K_1$ and $g: M_2 \to K_2$. Define $h: M \to N$ by $h(a,b) = (f(a), g(b))$. It can be easily checked that *h* is a monomorphism and hence *M* is small compressible.

2.8 Corollary:

The direct sum of a finite family of small compressible modulesM_i,$i = 1,2, \ldots, n$ with $\sum_{i=1}^{n} annM_i = R$ is also small compressible.

2.9Definition[8]:

An *R*-module *M* is called small prime if $annM = annN$ for each non-zero small submodule *N* of *M*.

2.10Definition[8]:

A proper submodule *N* of an *R*-module *M* is called small primesubmodule if and only if whenever $r \in R$ and $x \in M$ with $(x) \ll M$ and $rx \in N$ implies either $x \in N$ or $r \in [N:M]$.

2.11Definition[5]:

An R-module M is called small uniform if every non-zero small submodule of M is essential in M.

2.12 Proposition[5]:

A finitely generated module *M* is small compressible if and only if *M* is small uniform and small prime.

We introduce in the following theorem some characterizations of small compressible modules.

2.13 Theorem:

Let M be an R-module. Then the following statements are equivalent:

(1) M is small compressible.

(2) M is isomorphic to an R-module of the form A/P for some prime ideal P of R and an ideal A of R containing P properly.

(3) M is isomorphic to a non-zero submodule of a finitely generated small uniform, small prime R-module.

Proof: (1)⟹(2) Let $0 \neq m \in M$ and $Rm \ll M$. Then Rm is small compressible by proposition (2.3). Therefore Rm is small prime by proposition (2.12). By (1), there is a monomorphism, say $f: M \to Rm$ and hence M is isomorphic to a submodule of Rm. On the other hand, $Rm \simeq R/ann(m)$ and M is small prime implies that $ann(m)$ is a prime ideal in R[8]. Put $ann(m) = p$. Then $M \simeq A/P$ where A is an ideal of R contains P properly and P is a prime ideal of R.

(2)⟹(3) By (2), $M \simeq A/P$ for some prime ideal P of R and an ideal A of R containing P properly, so A/P is a non-zero submodule of R/P. R/P is finitely generated R-module and R/P is a small prime R-module (since R/P is an integral domain).Also R/P is a uniform R-module and hence small uniform, hence (3) follows.

(3) ⟹(1) By (3), M is isomorphic to a non-zero submodule of a finitely generated small uniform and small prime R-module, say $\grave{M}$, so $\grave{M}$ is small compressible R-module by proposition (2.12). Hence M is also small compressible R-module by proposition (2.5) which proves (1).

In the following theorem we give a necessary condition for a quotient module to be small compressible.

2.14 Theorem:

Let N be a proper submodule of an R-module M such that $[N:M] \not\supseteq [K:M]$ for each submodule K of M containing N properly. If M/N is small compressible, then N is small prime submodule of M. The converse holds if every cyclic submodule of M is small in M.

Proof: Let $r \in R, x \in M, (x) \ll M$ and $rx \in N$. Suppose that $x \notin N$. Then$N \subsetneq N + (x)$.We claim that $\frac{N+(x)}{N} \ll \frac{M}{N}$. Suppose that $\frac{N+(x)}{N} + \frac{L}{N} = \frac{M}{N}$, for some submodule L of M containing N. Hence $\frac{N+(x)+L}{N} = \frac{M}{N}$so $\frac{(x)+L}{N} = \frac{M}{N}$implies that $(x) + L = M$. But $(x) \ll M$ by hypothesis, therefore $L = M$ and $\frac{L}{N} = \frac{M}{N}$ which means that $\frac{N+(x)}{N} \ll \frac{M}{N}$ .Therefore there exists a monomorphism, say $f: \frac{M}{N} \to \frac{N+(x)}{N}$ (since $\frac{M}{N}$ is small compressible by hypothesis). It can be easily checked that for all $r \in R, rf\left(\frac{M}{N}\right) = f\left(\frac{rM}{N}\right) = N$), and since f is a monomorphism therefore $\frac{rM}{N} = N$ and hence $rM \subseteq N$ implies that $r \in [N:M]$ which proves that N is small prime.

Conversely, Assume that N is a small prime submodule of M. We have to show that M/N is small compressible. Let $0 \neq L/N \ll M/N$. Then $[N:M] \not\supseteq [L:M]$ (by hypothesis) and hence there exists $t \in [L:M]$ and $t \notin [N:M]$. Define $f: M/N \to L/N$ by $f(m + N) = tm + N$ for all $m \in M$. Clearly, f is a homomorphism. To prove f is a monomorphism. Suppose that $f(m_1 + N) = f(m_2 + N)$ with $m_1, m_2 \in M$. Then $tm_1 - tm_2 = t(m_1 - m_2) \in N$. But by hypothesis $(m_1 - m_2) \ll M$ and N is small prime submodule of M, moreover $t \notin [N:M]$, therefore $m_1 - m_2 \in N$ and hence $m_1 + N = m_2 + N$. Hence f is a monomorphism which completes the proof.

The following are some consequences of theorem (2.14)

2.15 Corollary:

Let M be an R-module such that $annM \not\supseteq [K{:}M]$ for all non-zero submodule K of M and every cyclic submodule of M is small in M. Then M is small compressible.

2.16 Corollary:

Let M be an R-module such that $annM \not\supseteq [K{:}M]$ for each submodule K of M and every cyclic submodule of M is small in M. Then M is small prime if and only if M is small compressible.

2.17 Corollary:

Let M be a multiplication R-module, N be a proper submodule of M and every cyclic submodule of M is small in M. Then M/N is small compressible is and only if N is small prime submodule of M.

Proof: As M is a multiplication module, then $[N{:}M] \not\supseteq [K{:}M]$for all submodule K of M containing N properly. So according to theorem (2.14) the result follows.

2.18 Corollary:

Let I be a proper ideal of a ring R such that every principal ideal of R is small in R. Then R/I is small compressible if and only if I is a small prime ideal of R.

2.19 Proposition[5]:

Let M be a faithful finitely generated multiplication R-module. Then M is small compressible module if and only if R is small compressible ring.

2.20 Definition (2.1.22)[5]:

A small compressible module M is called small critically compressible if M cannot be embedded in any proper quotient module M/N with $0 \neq N \ll M$.

2.21 Proposition:

A non-zero small submodule of a small critically compressible module is also small critically compressible.

Proof: Let M be a small critically compressible module and $0 \neq N \ll M$. Then by proposition (2.3)N is small compressible. Let $0 \neq H \ll N$. Then $H \ll M$ and $N/H \ll M/H$ [6].Suppose that there exists a monomorphism say $\alpha{:}N \to N/H$. But M is small compressible implies that there is a monomorphism say $f{:}M \to N$. Then the composition M$\xrightarrow{f} N \xrightarrow{\alpha} N/H \xrightarrow{i} M/H$gives a monomorphism from M into M/H which is a contradiction. Therefore N is small critically compressible.

2.22Proposition:

A direct summand of a small critically compressible is small critically compressible.

Proof: Let $M = A \oplus B$ be a small critically compressible module. Then M is small compressible and by proposition (2.4), A is also small compressible. Let $0 \neq K \ll A$. Then $K \simeq K \oplus 0 \ll M$.Let $f{:}M \to K$ be a monomorphism, and suppose that there is a monomorphism say, $g: A \to A / K$. Then the composition $M \xrightarrow{f} K \xrightarrow{i} A \xrightarrow{g} A / K \xrightarrow{j} M / K$ is a monomorphism (where i and j are the inclusion homomorphisms). Therefore a contradiction. Hence A is small critically compressible.

We introduce the following concept:

2.23 Definition:

A small partial endomorphism of a module M is a homomorphism from a small submodule of M into M.

2.24 Proposition:

Let M be a small critically compressible module. Then every non-zero small partial endomorphism of M is a monomorphism.

Proof: Let $0 \neq N \ll M$and $f: N \to M$ be a non-zero small partial endomorphism. Then $N/kerf \simeq f(N)$ and $0 \neq f(N) \ll M$ [6]. Let $\varphi: N/kerf \to f(N)$ be an isomorphism. But M is small critically compressible (by hypothesis) implies that there exists a monomorphism, say $g: M \to f(N)$, so the composition $M \xrightarrow{g} f(N) \xrightarrow{\varphi^{-1}} N/kerf \xrightarrow{i} M/kerf$ is a monomorphism and $kerf \leq N \ll M$ gives $kerf \ll M$. Thus M is embedded in $M/kerf$ which is a contradiction then, $kerf = 0$, so f is a monomorphism.

The following proposition is a partial converse of proposition (2.24)

2.25 Proposition:

Let M be a small compressible module such that the quotient of every submodule of M by a small submodule is small. If every small partial endomorphism of M is a monomorphism, then M is small critically compressible.

Proof: Suppose that M is not small critically compressible then there is a non-zero small submodule N of M and a monomorphism $f: M \to M/N$. Therefore M is isomorphic to a submodule, say K/N of M/N with K is a submodule of M containing N. By hypothesis $K/N \ll M/N$ and since $N \ll M$ implies $K \ll M$[6]. Hence the composition $K \xrightarrow{\pi} K/N \xrightarrow{\varphi^{-1}} M$ (where $\varphi: M \to K/N$ is an isomorphism) is a monomorphism (by hypothesis) and hence $0 = \ker(\varphi^{-1}\pi) = ker\pi = N$ which is a contradiction, therefore M is small critically compressible.

3. SMALL RETRACTABLE MODULES

In this section we study the concept of small retractable modules in some details.

3.1 Definition[5]:

An R-module M is called small retractable if $\mathrm{Hom}_R(M, N) \neq 0$ for each non-zero small submodule N of M.

A ring R is called small retractable if the R-module R is small retractable. That is $\mathrm{Hom}_R(R, I) \neq 0$ for each non-zero small ideal I of R.

3.2 Examples and Remarks:

(1) Every retractable module is small retractable and the converse is not always hold. Consider the following example:

Let $S = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in R\}$ where R be a commutative ring with identity. S is a ring with identity with respect to addition and multiplication of matrices. The non-zero ideals of S are:

$$I = S, I = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R\right\}, I = \left\{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} : a, c \in R\right\}, I = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in R\right\} \text{ or } I = \{\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : c \in R\}.$$

In each of these cases one can easily define a non-zero homomorphism from S to I, which means that S is a retractable S-module.

Now, let $I = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R\right\}$.we claim that I is not a retractable submodule of S.

Note that $I = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} S$ and $\begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$ is an idempotent element and hence I is an idempotent ideal.

Let $J = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in R\}$. J is a subideal of I and $JI = 0$ suppose that there is a homomorphism, say $f: I \to J$.

Then $f(I) = f(I^2) = f(I)I \subseteq JI = 0$ and hence $f(I) = 0$, that means $f = 0$,therefore $Hom(I, J) = 0$, hence I is not retractable.on the other hand the only small submodule of I is the zero submodule, hence I is small retractable.

(2) If M is a hollow module, then M is retractable if and only if M is small retractable.

(3) The Z-module Q is not small retractable since $Z \ll Q$ but $Hom_R(Q, Z)$= 0.

(4) Every integral domain is a small retractable ring but not conversely, for instance Z_6 as a Z_6-module is small retractable but $Z_6$is not an integral domain.

(5) Every semisimple module is small retractable, however the converse is not true in general, for example Z is small retractable Z-module but it is not semisimple.

(6) Every module over a semisimple ring is small retractable.

(7) Every small compressible module is small retractable and the converse is not true in general, for example the Z-module, $Z_{24}$is small retractable but not small compressible since $\{\bar{0}, 1\bar{2}\}$ is the only small submodule in Z_{24} and $f: Z_{24} \to \{\bar{0}, \overline{12}\}$ such that $f(\bar{x}) = 12\bar{x}$ for all $\bar{x} \in Z_{24}$ is a homomorphism which is not monomorphism.

(8) Let M be an R-module. Then M is a small retractable R-module if and only if M is a small retractable $R/annM$-module.

3.3 Proposition:

Let M be an R-module such that $End_R(M)$ is a Boolean ring. If M is small retractable, then every non-zero small submodule of M is also small retractable.

Proof: Let N be a small submodule of M and K be a small submodule of N. Then $Hom_R(M, K) \neq 0$. Let $f: M \to K$ be a non-zero homomorphism. Hence $fi: N \to K$ is a homomorphism where $i: N \to M$ is the inclusion homomorphism. We claim that $fi \neq 0$, Suppose that $fi = 0$, then $(fi)(N) = 0 = f(N)$,so $N \subseteq Kerf$ and hence $K \subseteq Kerf$, which implies that $f(M) \subseteq Kerf$ therefore $f(f(M)) = 0$. Let $j: K \to M$ be the inclusion homomorphism. Then $jf \in End_R(M)$and $jf(M) = f(M)$ *but* $(jf)^2(M) = (jf)(jf)(M) = jf(f(M)) = j(f(f(M)) = j(0) =$

0, and$(jf)^2(M) = (jf)(M)$since $End_R(M)$ is a Boolean ring.Hence $j(f(M)) = f(M) = 0$. Therefore $f = 0$ which is a contradiction, thus $fi \neq 0$, therefore N is small retractable.

3.4 Proposition:

Let $M_1$and M_2be two isomorphic R-modules. Then M_1 is small retractable if and only if $M_2$is small retractable.

Proof: Assume that $M_1$is retractable and let $\varphi : M_1 \to M_2$ be an isomorphism. Let N be a mall submodule of M_2. Then $\varphi^{-1}(N)$Be a non-zero small submodule of M_1 Put $K = \varphi^{-1}(N)$. Let $f: M_1 \to K$be a non-zero homomorphism and let$g = \varphi\big|_K$ then $g: K \to M_2$ is a homomorphism and $g(k) = \varphi(\varphi^{-1}(N)) = N$, hence $g: K \to N$ is a homomorphism. Now, we have the composition $M_2 \xrightarrow{\varphi^{-1}} M_1 \xrightarrow{f} K \xrightarrow{g} N$. Let $h = gf\varphi^{-1}$, then $h \in Hom\left(M_2, N\right)$.If $h = 0$,then $0 = g(f\left(\varphi^{-1}\left(M_2\right)\right) = g(f\left(M_1\right))$implies that $f\left(M_1\right) \subseteq Kerg \subseteq Ker\varphi = 0$.Thus $f\left(M_1\right) = 0$, which is a contradiction. Therefore $Hom_R\ (M_2, N) \neq 0$ which is what we wanted.

3.5 Corollary:

If R is a small retractable ring, then every faithful cyclic R-module is also small retractable.

3.6 Remark:

A direct summand (and a homomorphic image, or a quotient module) of a small retractable module may not be small retractable in general.

For example, the Z-module $Z \oplus Z_{p^\infty}$ is small retractable, however $Z_{p^\infty}$is not small retractable, $M/Z \simeq Z_{p^\infty}$is not small retractable since $Z_{p^\infty}$is a hollow Z-module.

3.7 Proposition:

If M_1 and $M_2$are small retractable modules such that $annM_1$ +ann$M_2 = R$ then $M_1 \oplus M_2$is also small retractable.

Proof: Let $0 \neq K \ll M_1 \oplus M_2$. As $annM_1 + annM_2 = R$by [9]gives $K = N_1 \oplus N_2$ with $N_1 \leq M_1$ and $N_2 \leq M_2$.But $N_1 \oplus N_2 \ll M_1 \oplus M_2$ implies $N_1 \ll M_1$and $N_2 \ll M_2$ [6].Therefore$Hom(M_1, N_1) \neq 0$and $Hom(M_2, N_2) \neq 0$. Let $0 \neq f: M_1 \to N_1$ and $0 \neq g: M_2, \to N_2$.Define $h: M_1 \oplus M_2 \to N_1 \oplus N_2$ by $h(m_1, m_2) = (f(m_1), g(m_2))$ clearly h is a homomorphism. If $h = 0$, then $h(m_1, m_2) = 0$ for all $m_1 \in M_1$,$m_2 \in M_2$,so $f(m_1) = 0$ and $g(m_2) = 0$ for all $m_1 \in M_1$,$m_2 \in M_2$, which is a contradiction since $f \neq 0$ and $g \neq 0$. Therefore $Hom(M_1 \oplus M_2, K) \neq 0$.

3.8 Proposition:

Let M be a small retractable module. If every non-zero submodule of M contains a non-zero small submodule then M is retractable.

Proof: Let $0 \neq N \leq M$. By hypothesis N contains a non-zero small submodule. Let $0 \neq K \ll N$. Then $K \ll M$ [4].Hence $Hom(M, K) \neq 0$ (since M is small retractable), and therefore $Hom(M, N) \neq 0$ so M is retractable.

As it was mentionedthat every small compressible module is small retractable and the converse need not be true in general, we recall in the following results that the converse holds under certain conditions:

3.9 Proposition:

Let M be a small retractable R-module. If every non-zero endomorphism of M is a monomorphism, then every non-zero element of $Hom(M,N)$ is a monomorphism from any non-zero small submodule *N* of *M*.

Proof: Let $0 \neq N \ll M$ and let $f: M \to N$ be a non-zero homomorphism. Then $if \in End(M)$ and $if \neq 0$. For if $if = 0$, then $if(M) = f(M) = 0$ implies $f = 0$ which is a contradiction. Hence $0 \neq if \in End(M)$ and by hypothesis if is a monomorphism which gives that *f* is a monomorphism.

3.10 Corollary:

Let *M* be a small retractable module such that every non-zero endomorphism of *M* is a monomorphism then *M* is small compressible.

3.11 Corollary[5]:

Let *M* be a small retractable module. If *M* is quasi-Dedekind, then *M* is small compressible.

3.12 Corollary:

Let *M* be a finitely generated module such that every non-zero endomorphism of *M* is a monomorphism then *M* is small retractable if and only if *M* is small prime and small uniform.

Proof: From corollary (3.10), *M* is small compressible and according to proposition (2.12), the result follows.

3.13 Corollary:

Let *M* be a small retractable quasi-Dedekind module. Then *M* is S-monoform if and only if each non-zero small submodule of *M* is quasi-Dedekind.

Proof: By corollary (3.11), *M* is small compressible and by [5], *M* is *S*-monoform.

From corollary (3.13) and theorem (2.13) we get:

3.14 Corollary:

If *M* is small retractable quasi-Dedekind module then the following statements are equivalent:

(1) *M* is small compressible.

(2) *M* isomorphic to an *R*-module of the form *A/P* where *P* is a prime ideal of *R* and *A* is an ideal of *R* containing *P*.

(3) *M* is isomorphic to a non-zero submodule of a finitely generated small uniform and small prime *R*-module.

4. SOME CHARACTERIZATIONS OF SMALL RETRACTABLE MODULES

We shall introduce some characterizations of small retractable modules

4.1 Proposition:

An R-module M is called small retractable if and only if there exists $0 \neq f \in End_R(M)$ such that $Im\, f \subseteq N$ for each non-zero small submodule N of M.

Proof: ($\Rightarrow$) Suppose that M is small retractable. Let $0 \neq N \ll M$. Then $Hom_R(M, N) \neq 0$. Let $g: M \to N$ be a non-zero homomorphism and $f = ig$ where $i: N \to M$ be the inclusion homomorphism, then $f \in End_R(M)$ and $f \neq 0$ since $f \neq 0$ and i is a monomorphism. Clearly, $f(N) = g(N) \subseteq N$.

($\Leftarrow$) Let $0 \neq N \ll M$. By hypothesis, there exists a non-zero endomorphism $f: M \to M$ and $f(M) \subseteq N$. Therefore $f: M \to N$ is a non-zero homomorphism this completes the proof.

4.2 Proposition:

An R-module M is small retractable if and only if for each $0 \neq x \in M$ with $Rx \ll M$, $Hom_R(M, Rx) \neq 0$.

Proof: ($\Rightarrow$) Is obvious.

($\Leftarrow$) To prove M is small retractable. Let $0 \neq N \ll M$ and let $0 \neq x \in N$, then $Rx \ll N$, so by hypothesis, $Hom(M, Rx) \neq 0$, which implies that $(M, N) \neq 0$ and therefore M is small retractable.

4.3 Proposition:

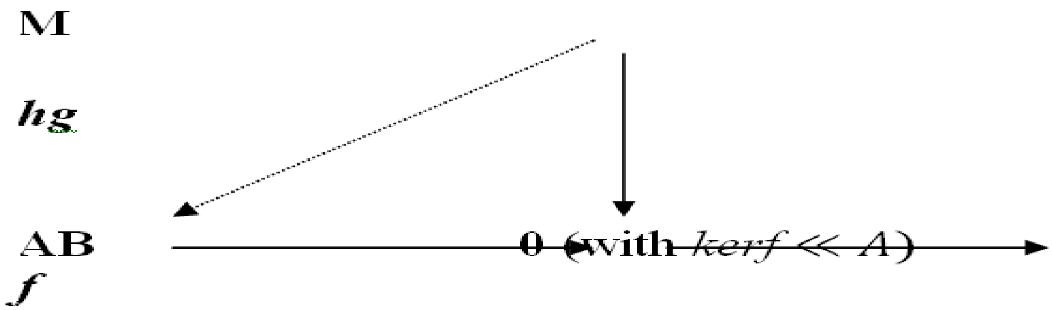
Let M be a fully invariant R-module such that $f(M)$ is a direct summand of M for each $f \in End_R(M)$. Then M is small retractable if and only if there exists $0 \neq f \in End_R(M)$ such that $f(M)$ is small retractable.

Proof:($\Rightarrow$) Let $i_M$be the identity endomorphism of M then $i_M(M) = M$ is small retractable.

($\Leftarrow$) To prove M is small retractable. Let $0 \neq N \ll M$. By hypothesis there is a non-zero homomorphism $f: M \to M$ and $f(M)$ is small retractable. Since $N \ll M$, then $f(N) \ll M$[6], but $f(N) \leq f(M) \leq M$ and $f(M)$ is a direct summand of M (by hypothesis) implies that $f(N) \ll f(M)$[6] . As $f(M)$ is small retractable, so there is a non-zero homomorphism $g: f(M) \to f(N)$. But $f(N) \subseteq N$ since N is invariant therefore the composition $M \xrightarrow{f} f(M) \xrightarrow{g} f(N) \xrightarrow{i} N$ gives $igf \in Hom(M, N)$ and $igf \neq 0$, for if $igf = 0$, then $0 = igf(M) = gf(M)$ implies $g = 0$ which is a contradiction. Therefore M is small retractable.

4.4 Definition[2]:

An R-module M is called small projective if for each small epimorphism$f: A \to B$(where A and B are any two R-modules) and for any homomorphism $g: M \to B$ there exists a homomorphism $h: M \to A$ such that $fh = g$. That is the following diagram is commutative.



where an epimorphism$f: A \to B$ is called small epimorphism provided that $kerf \ll A$[4].

4.5 Definition[3]:

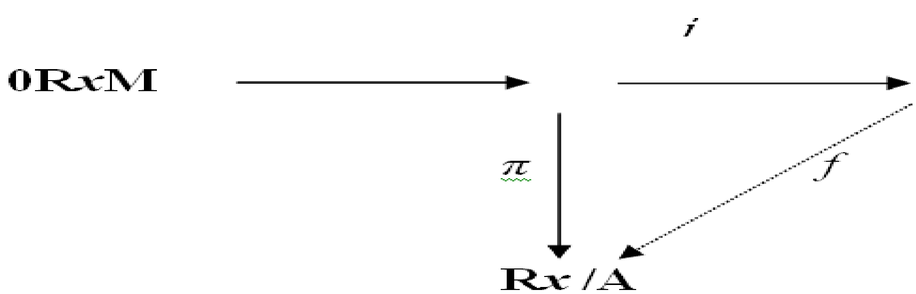
A ring R is called V-ring if every simple R-module is injective.

4.6 Proposition:

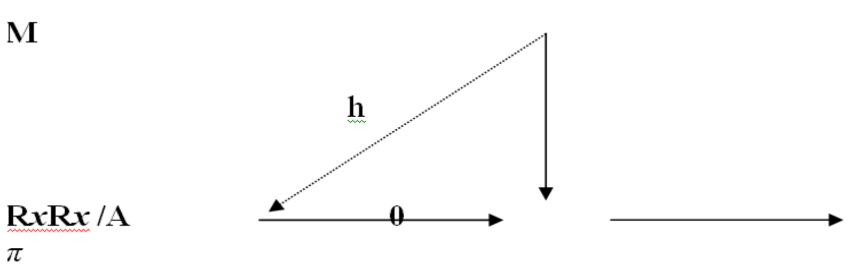
If R is a V-ring (or a von-Neumann regular ring), then every small projective R-module is small retractable.

Proof: Let M be a small projective R-module. Let $0 \neq x \in M$ such that $Rx \ll M$. We have to show that $Hom(M, Rx) \neq 0$. Let A be a maximal submodule of Rx. Then Rx/A is a simple R-module and hence Rx/A is injective R-module (since R is a V-ring).

Consider the following diagram:



Since Rx/A is injective implies that there exists $f: M \to Rx/A$ such that $fi = \pi$. Note that $kerf = A \leq Rx \ll M$, so $kerf \ll M$ and M being small projective implies that there exists a homomorphism $h: M \to Rx$ which makes the following diagram commutative



That is $\pi h = f$. We get $h \in Hom(M, Rx)$.It is left to show that $h \neq 0$. If $h = 0$, then $h(M) = 0$ and $A = \pi(0) = f(M)$. On the other hand $fi = \pi$ gives $fi(Rx) = \pi(Rx) = Rx + A$. Thus $f(Rx) = Rx + A \subseteq A$. Therefore $Rx \subseteq A$ implies $A = Rx$ which is a contradiction since A is a maximal submodule of Rx, and hence $h \neq 0$ which proves that M is small retractable.

4.7 Definition[2]:

A ring R is called cosemisimple if $Rad(M) = 0$, for each R-module M. where $Rad(M) =$ the sum of all small submodules of M.

4.8 Proposition[1]:

A ring R is cosemisimple if and only if every R-module is small projective.

The following result follows directly from propositions (4.7) and (4.8)

4.9 Corollary:

If R is a cosemisimple V-ring, then every R-module is small retractable.

A relation between small uniform module and small retractable module is discussed under, certain conditions in the following proposition:

4.10 Proposition:

Let R be an integral domain. Then every faithful finitely generated small uniform R-module is small retractable

Proof: Let M be a finitely generated small uniform R-module. Then $M = Rx_1 + Rx_2 + \cdots + Rx_n$ where $x_i \in M \forall i = 1,2, \ldots\ldots, n$. Let $0 \neq N \ll M$. Then q, $\ldots\ldots, n$ there exists $t_i \in R, t_i \neq 0$ and $0 \neq t_i \; x_i \in N$[7]. Let $t = t_1 t_2 \ldots\ldots . t_n$. Then $t \neq 0$ and $0 \neq t_i \; x_i \in N$ for all $i = 1,2, \ldots\ldots, n$ and for each $m \in M$, $m = \sum_{i=1}^{n} r_i \, x_i$ with$r_i \in R \; \forall i = 1,2, \ldots\ldots, n$. and $tm = \sum_{i=1}^{n} t \, (r_i \, x_i) = \sum_{i=1}^{n} r_i \, (t \, x_i)$ and hence $tm \in N, \forall m \in M$. So we can define $f: M \to N$ by $f(m) = tm \quad \forall m \in M$. Clearly f is anon-zero homomorphism, hence $Hom(M, N) \neq 0$, for if $f = 0$, then $tm = 0$ for all $m \in M$ implies $t = 0$ (since M is faithful), but $t \neq 0$ therefore a contradiction. Hence M is retractable.

4.11 Proposition:

Let R be a small retractable ring and M be a faithful finitely generated multiplication R-module. Then M is small retractable.

Proof: Let $0 \neq N \ll M$. Then $N = IM$ for some non-zero ideal I of R (since M is multiplication R-module). But $N \ll M$ and M is a faithful finitely generated multiplication R-module implies that $I \ll R$ [6] therefore $Hom(R, I) \neq 0$ (since R is small retractable). Let $f: R \to I$ be a non-zero homomorphism. Put $f(1) = a$ for some $a \in I$ and $a \neq 0$. Define $g: M \to N$ by $g(m) = am$ for all $m \in M$. It can be easily checked that g is a well-defined homomorphism, if $g = 0$, then

$am = 0$ for all $m \in M$ and therefore $a \in ann(M)$, hence $a = 0$ (since M is faithful) but $a \neq 0$, therefore a contradiction and hence $Hom(M, N) \neq 0$. Therefore M is small retractable.

4.12 Remark:

The ring Z is small retractable but the Z-module Q is not small retractable, in fact Q is not finitely generated multiplication Z-module. This means that these two conditions cannot be dropped in the proposition (4.11).

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